

DEFORMATION THEORY FOR HOLOMORPHIC FOLIATIONS

T. DUCHAMP & M. KALKA

Introduction

In this paper we consider deformations of holomorphic foliations on compact manifolds. By a holomorphic foliation we mean a foliation given by local submersions $f_\alpha: U_\alpha \rightarrow R^{2q}$ which patch together via maps $\varphi_{\alpha\beta}: R^{2q} \rightarrow R^{2q}$, which are local biholomorphisms when R^{2q} is identified with C^q .

For \mathcal{F} a holomorphic foliation on a manifold M , we show that the infinitesimal deformations of \mathcal{F} correspond to elements of $H^1(M, \theta_{\mathcal{F}})$, where $\theta_{\mathcal{F}}$ is the sheaf of germs of holomorphic vector fields on the normal bundle of \mathcal{F} which are constant on the leaves of \mathcal{F} . For example, if \mathcal{F} is given by the fibers of a submersion onto a complex manifold, then $\theta_{\mathcal{F}}$ is the pull-back of the sheaf θ_N of germs of holomorphic vector fields on the image. By constructing explicitly a resolution of $\theta_{\mathcal{F}}$ by an elliptic complex (E_Q^{**}, d_Q) we show that $H^*(M, \theta_{\mathcal{F}})$ is finite dimensional.

Resolutions of the sheaf of sections of the normal bundle of a C^∞ -foliation which are constant on leaves have appeared in the works of Hamilton [4], Heitsch [5], Kamber-Tondeur [6], Mostow [9] and Vaisman [15]. Also in the case where M is a complex manifold and the submersions f_α are holomorphic, Heitsch has constructed a resolution of the sheaf $\theta_{\mathcal{F}}$ and shown that its cohomology groups are finite dimensional. Our resolution is different from his and applies to the case where M is only a smooth manifold. Of course the general theory of pseudogroup structures on manifolds developed by Spencer [13] applies to the case of holomorphic foliations on smooth manifolds. However, the relevant pseudogroup is neither elliptic nor complex; hence the Spencer complex associated to such a foliation does not directly lead to finite dimensionality results and the theory of elliptic complexes does not apply to it.

Having constructed a resolution of $\theta_{\mathcal{F}}$, we then show how to extend Kuranishi's theorem on the existence of a locally complete finite dimensional holomorphic family for complex structures close to a given complex structure,

to holomorphic foliations. In order to do this, it is necessary to define a bracket operation $[\ , \]_Q: E_Q^{*r} \times E_Q^{*s} \rightarrow E_Q^{*(r+s)}$ with certain nice properties (2.11–2.14). In general we cannot do this. However, if we assume that there is a C^∞ -foliation \mathcal{F}^\perp transverse to the foliation \mathcal{F} , then such a bracket can be defined. Distributions near the tangent bundle of \mathcal{F} are given by elements of $\Gamma(E_Q^{*1})$, and the integrability condition in the complex Frobenius theorem takes the form $d_Q - [\ , \]_Q = 0$. The operator d_Q is just the sum of the Dolbeaut operator in the holomorphic directions normal to \mathcal{F} and the de Rham operator in directions parallel to \mathcal{F} . Note that the transverse foliation \mathcal{F}^\perp allows us to consider all bundles as sub-bundles of the complexified tangent bundle of M or its dual. The proof proceeds exactly as in Kuranishi [8], only the bundles and the operators have been changed. In fact Kuranishi's theorem is a special case of our theorem, where the leaves of the foliation are the points of M .

We then consider the problem of computing $H^1(M, \underline{\theta}_{\mathcal{F}})$. In particular, we consider the case where \mathcal{F} is given by a fibration $M \xrightarrow{f} N$ with N a complex manifold and with fiber S . We show that if $H_{DR}^1(S) = 0$ and $H^1(N, \underline{\theta}_N) = 0$, then $H^1(M, \underline{\theta}_{\mathcal{F}}) = 0$, where $\underline{\theta}_N$ is the sheaf of germs of holomorphic vector fields on N . If the structure group of the fibration is discrete, this implies that there are no small deformations of \mathcal{F} , up to equivalence. This should be compared with Hamilton's result [4] that if \mathcal{F} is a C^∞ -Hausdorff foliation with $H_{DR}^1(L) = 0$, where L is the generic leaf of \mathcal{F} , then \mathcal{F} is structurally stable.

The paper is organized as follows: In §1 we describe the relevant elliptic complexes and define the operator d_Q . In §2 we define the bracket operator $[\ , \]_Q$ and derive the partial differential equation which is the integrability condition in the complex Frobenius theorem. In §3 we solve this equation and prove Kuranishi's theorem. In §4 we compute $H^1(M, \underline{\theta}_{\mathcal{F}})$ in certain cases. The techniques of §4 are similar to those of Mostow [9]. The main results of this paper are Theorems 1.27, 2.4 and 3.1.

We will use the following notational conventions: Latin subscripts (or superscripts) will run from 1 to p , whereas Greek subscripts (or superscripts) will run from 1 to q where $n = p + 2q$. Also if B is a vector bundle over M , then we will denote by \underline{B} the sheaf of germs of sections of B . If \underline{S} is a sheaf, the space of global sections of \underline{S} will be denoted by $\Gamma(\underline{S})$. If B is a bundle, the space of its global sections will be denoted by $\Gamma(B)$. We will use the Einstein summation conventions.

1. Elliptic complexes associated to a holomorphic foliation

Let M be an n -dimensional C^∞ -manifold. We investigate here holomorphic

foliations on M close to a fixed holomorphic foliation. We recall that a (real) codimension- $2q$ holomorphic foliation \mathcal{F} is given by an open cover $\{U_\alpha\}_{\alpha \in A}$ of M , a collection of submersions $f_\alpha: U_\alpha \rightarrow \mathbb{C}^q$, and associated maps $\varphi_{\alpha\beta}^x$ for each $x \in U_\alpha \cap U_\beta$, which are local biholomorphic maps and satisfy $f_\beta(y) = \varphi_{\beta\alpha}^x \circ f_\alpha(y)$ for y near x . For the foliation to be global it is necessary that the collection $\{\varphi_{\alpha\beta}^x\}_{\alpha, \beta \in A}$ satisfy the cocycle condition $\varphi_{\alpha\gamma}^x = \varphi_{\alpha\beta}^x \circ \varphi_{\beta\gamma}^x$ for all α, β, γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ and for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$. A collection $\{U_\alpha, f_\alpha, \{\varphi_{\alpha\beta}^x\}_{x \in U_\alpha \cap U_\beta}\}$ which is maximal with respect to the above properties is called a Haefliger cocycle with coefficients in $\Gamma_{\mathbb{C}}^q$, the pseudogroup of local biholomorphisms of \mathbb{C}^q . We set $\mathcal{H}_{\mathcal{F}} = \{f_\alpha: U_\alpha \rightarrow \mathbb{C}^q, \varphi_{\alpha\beta}^x \in \Gamma_{\mathbb{C}}^q, \alpha, \beta \in A\}$.

Associated to a codimension $-2q$ holomorphic foliation \mathcal{F} is its tangent bundle L . Set $p = \dim L_x = n - 2q$. The normal bundle of \mathcal{F} is the bundle $Q = TM/L$. There is an almost complex structure on Q obtained by lifting the standard complex structure on \mathbb{C}^q to Q via the submersions f_α . We recall that Q is locally representable as the pull-back, along f_α , of the tangent bundle of \mathbb{C}^q .

The almost complex structure on Q induces a splitting of the complexified normal bundle in the standard way

$$(1.1) \quad Q^{\mathbb{C}} = Q^{(1,0)} \oplus Q^{(0,1)}.$$

We will need several short exact sequences of vector bundles associated to the foliation. Consider first the sequence defining Q :

$$(1.2) \quad 0 \rightarrow L \xrightarrow{i_L} T_M \xrightarrow{\pi} Q \rightarrow 0.$$

Because of (1.1) we have an exact sequence of complex vector bundles

$$(1.3) \quad 0 \rightarrow E \xrightarrow{i_E} T_M^{\mathbb{C}} \xrightarrow{\pi^{(1,0)}} Q^{(1,0)} \rightarrow 0,$$

where $\pi^{(1,0)}$ is defined to be the composition

$$T_M^{\mathbb{C}} \xrightarrow{\pi^{\mathbb{C}}} Q^{\mathbb{C}} \rightarrow Q^{(1,0)}$$

and $E = \text{Ker } \pi^{(1,0)}$. We note that there is a noncanonical isomorphism $E \simeq L^{\mathbb{C}} \oplus Q^{(0,1)}$. We also consider the exact sequence of real vector bundles

$$(1.4) \quad 0 \rightarrow L \xrightarrow{i_L} T_M \xrightarrow{\tau} Q^{(1,0)} \rightarrow 0.$$

The map τ in (1.4) is the composition

$$T_M \rightarrow T_M^{\mathbb{C}} \xrightarrow{\pi^{(1,0)}} Q^{(1,0)}.$$

To check that (1.4) is exact we need to check that $\text{Ker } \tau = L$ and that τ is surjective. We do this in local coordinates. Let $U \subseteq M$ be open and let f :

$U \rightarrow \mathbf{C}^q$ be a submersion in $\mathcal{K}_{\mathfrak{F}}$. We can consider U to be contained in

$$\mathbf{R}^p \times \mathbf{C}^q = \{(x, z) | x = (x^1, \dots, x^p), z = (z^{p+1}, \dots, z^n)\},$$

and f to be the map $(x, z) \rightarrow z$. Such a coordinate system is said to be adapted to the holomorphic foliation \mathfrak{F} . Let $z^\alpha = u^\alpha + iv^\alpha$. Then T_M is spanned by the vector fields $\partial/\partial x^j, \partial/\partial u^\alpha, \partial/\partial v^\alpha$. Also $Q^{\mathbf{C}}$ is spanned by the vector fields

$$(1.5) \quad \left[\frac{\partial}{\partial z^\alpha} \right] = \frac{1}{2} \left[\frac{\partial}{\partial u^\alpha} - i \frac{\partial}{\partial v^\alpha} \right], \quad \left[\frac{\partial}{\partial \bar{z}^\beta} \right] = \frac{1}{2} \left[\frac{\partial}{\partial u^\beta} + i \frac{\partial}{\partial v^\beta} \right],$$

and $Q^{(1,0)}$ is spanned by the vector fields $[\partial/\partial z^\alpha]$. Finally L is spanned by the vector fields $\partial/\partial x^\alpha$. We write $[\partial/\partial z^\alpha]$ for the equivalence class of $\partial/\partial z^\alpha$ under the projection $T_M^{\mathbf{C}} \rightarrow Q^{\mathbf{C}}$. Then for $X = X^j \partial/\partial x^j + U^\alpha \partial/\partial u^\alpha + V^\alpha \partial/\partial v^\alpha$ we have

$$(1.6) \quad \tau(X) = (U^\alpha + iV^\alpha) \left[\frac{\partial}{\partial z^\alpha} \right] \in Q^{(1,0)}.$$

It is now clear that τ is surjective and $\text{Ker } \tau = L$.

If $B \subseteq T_M^{\mathbf{C}}$ is a complex vector sub-bundle, we will denote its complex conjugate bundle by \bar{B} . With this definition we see that

$$(1.7) \quad T_M^{\mathbf{C}} = E + \bar{E}.$$

We will need the following version of the complex Frobenius theorem of Nirenberg [11].

Theorem 1.8. *Let $E \subseteq T_M^{\mathbf{C}}$ be a sub-bundle of $T_M^{\mathbf{C}}$ of complex codimension q with $E + \bar{E} = T_M^{\mathbf{C}}$. Let $Q^{(1,0)} \equiv T_M^{\mathbf{C}}/E$ so that $Q^{(1,0)*} \subseteq T_M^{\mathbf{C}*}$. Then the following conditions are equivalent:*

- (1) $[\underline{E}, \underline{E}] \subseteq \underline{E}$.
- (2) $d\underline{Q}^{(1,0)*} \wedge \underline{Q}_M^p \subseteq \underline{Q}^{(1,0)*} \wedge \underline{Q}_M^{p+1}$ for all $p \geq 0$.
- (3) \underline{E} and $Q^{(1,0)}$ are obtained from a codimension $-2q$ holomorphic foliation as in the above discussion.

Let $\rho: Q \rightarrow T_M$ be a splitting of (1.2). ρ induces a splitting of (1.3)

$$(1.9) \quad 0 \rightarrow E \xrightarrow[\varphi]{i_{\mathbf{C}}} T_M^{\mathbf{C}} \xrightarrow[\rho^{(1,0)}]{\pi^{(1,0)}} Q^{(1,0)} \rightarrow 0.$$

The splitting (1.9) allows us to define a one-one correspondence between distributions (a distribution will be used to mean a sub-bundle of the tangent bundle and not a generalized function) near E and the space $\text{Hom}_{\mathbf{C}}(E, Q^{(1,0)})$ in the following way. For $\varphi \in \text{Hom}_{\mathbf{C}}(E, Q^{(1,0)})$ let $E_\varphi \subseteq T_M^{\mathbf{C}}$ be defined by $E_\varphi = \{i_\varphi(X) = i_{\mathbf{C}}(X) + \rho^{1,0} \circ \varphi(X) | X \in E\}$. Conversely let $E' \subseteq T_M^{\mathbf{C}}$ be a

sub-bundle near E . Then the map

$$E' \subseteq T_M^{\mathbb{C}} \xrightarrow{\varphi} E$$

is an isomorphism. The inverse of this map is clearly of the form $X \mapsto i_{\mathbb{C}}(X) + \rho^{(1,0)} \circ \varphi(X)$ for a unique element $\varphi \in \text{Hom}(E, Q^{(1,0)})$.

Remark 1.10. For $\varphi \in \text{Hom}_{\mathbb{C}}(E, Q^{(1,0)})$ we have $E_{\varphi} + \bar{E}_{\varphi} = T_M^{\mathbb{C}}$.

Hence by Theorem (1.8) there is a one-to-one correspondence between holomorphic foliations near \mathcal{F} and the set

$$(1.11) \quad \text{Fol}(\mathcal{F}) = \{ \varphi \in \text{Hom}_{\mathbb{C}}(E, Q^{(1,0)}) \mid [E_{\varphi}, \bar{E}_{\varphi}] \subseteq E_{\varphi} \}.$$

We wish to characterize $\text{Fol}(\mathcal{F})$ as an analytic subspace of a neighborhood in the first cohomology group of a certain sheaf on M . In order to do this it is necessary to define this sheaf and to construct a resolution of it by an elliptic complex. This will enable us to use the theory of elliptic partial differential equations.

We have the following short exact sequence of sheaves

$$(1.12) \quad 0 \rightarrow Q^{(1,0)*} \wedge \underline{\Omega}_M^{s-1} \rightarrow \underline{\Omega}_M^s \xrightarrow{i_{\mathbb{C}}} E^{*s} \rightarrow 0,$$

where E^{*s} is the sheaf of local sections of the bundle $\Lambda^s E^*$. So (1.12) is the exact sequence induced by the exact sequence

$$(1.13) \quad 0 \rightarrow Q^{(1,0)*} \wedge \Lambda^{s-1} T_M^{\mathbb{C}} \rightarrow \Lambda^s T_M^{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} \Lambda^s E^* \rightarrow 0.$$

By Theorem 1.8 (2) we can define $\tilde{d}: Q^{(1,0)*} \wedge \underline{\Omega}_M^s \rightarrow Q^{(1,0)*} \wedge \underline{\Omega}_M^{s+1}$ as the restriction of the exterior derivative operator. Hence we can define $d_{\mathbb{C}}$ as the unique operator which makes the diagram

$$(1.14) \quad \begin{array}{ccccccc} 0 \rightarrow & \underline{Q}^{(1,0)*} & \wedge & \underline{\Omega}_M^{s-1} & \rightarrow & \underline{\Omega}_M^s & \xrightarrow{i_{\mathbb{C}}} & E^{*s} & \rightarrow & 0 \\ & & & \downarrow \tilde{d} & & & & \downarrow d & & \downarrow d_{\mathbb{C}} \\ 0 \rightarrow & \underline{Q}^{(1,0)*} & \wedge & \underline{\Omega}_M^s & \rightarrow & \underline{\Omega}_M^{s+1} & \xrightarrow{i_{\mathbb{C}}} & E^{*s+1} & \rightarrow & 0 \end{array}$$

commute. For $s = 0$, let $E^{*0} = \underline{C}_M^{\infty}$, the sheaf of local complex-valued C^{∞} -functions and let

$$(1.15) \quad d_{\mathbb{C}}: \underline{C}_M^{\infty} \xrightarrow{d} \underline{\Omega}_M^1 \xrightarrow{i_{\mathbb{C}}} E^{*1}.$$

In adapted coordinates $(x, z) = (x^1, \dots, x^p, z^{p+1}, \dots, z^n)$ the sheaf E^{*s} can be identified with the sheaf generated by the forms

$$[dx^1], \dots, [dx^p], [d\bar{z}^{p+1}], \dots, [d\bar{z}^n].$$

For $\varphi = \varphi_I [dx^{i_1} \wedge \dots \wedge dx^{i_s}] \wedge [d\bar{z}^{i_{s+1}} \wedge \dots \wedge d\bar{z}^{i_s}]$ in E^{*s} we have

$$(1.16) \quad d_{\mathbb{C}}\varphi = d_{\mathbb{C}}\varphi_I \wedge [dx^{I'}] \wedge [d\bar{z}^{I''}],$$

where $I = (I'; I'') = (i_1, \dots, i_t; i_{t+1}, \dots, i_s)$ and

$$(1.17) \quad d_e \varphi_I = \frac{\partial \varphi_I}{\partial x^j} [dx^j] + \frac{\partial \varphi_I}{\partial \bar{z}^\alpha} [d\bar{z}^\alpha],$$

$[dx^I]$ denoting the image of dx^I under i_E^* .

Let $\mathcal{O}_{\mathcal{F}} \subset \mathcal{C}_M^\infty$ be the subsheaf of smooth complex-valued functions which are locally lifts, via the submersions in $\mathcal{K}_{\mathcal{F}}$, of holomorphic functions on \mathbb{C}^q . Specifically, let $f: U \rightarrow \mathbb{C}^q$ be a submersion in $\mathcal{K}_{\mathcal{F}}$ and define

$$(1.18) \quad \Gamma(U, \mathcal{O}_{\mathcal{F}}) = f^* \mathcal{O}_{\mathbb{C}^q} = \{g \circ f \mid g \text{ is holomorphic on } f(U)\}.$$

Lemma 1.19. *The sequence*

$$(1.20) \quad 0 \rightarrow \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{C}_M^\infty \xrightarrow{d_e} \underline{E}^{*1} \xrightarrow{d_e} \underline{E}^{*2} \xrightarrow{d_e} \dots$$

is a resolution of the sheaf $\mathcal{O}_{\mathcal{F}}$.

Proof. We work in adapted coordinates. Note that if $d_e f = 0$ for $f \in \mathcal{C}_M^\infty$; then $\partial f / \partial x_j = 0$, $j = 1, \dots, p$. Hence, if $\pi: \mathbb{R}^p \times \mathbb{C}^q \rightarrow \mathbb{C}^q$ is the projection onto the second factor, we get $f = g \circ \pi$ where $g \in C^\infty(\mathbb{C}^q)$. Also $\partial g / \partial \bar{z}^\alpha = \partial f / \partial \bar{z}^\alpha \circ \pi = 0$ for $\alpha = 1, \dots, q$. Hence g is holomorphic and $f \in \mathcal{O}_{\mathcal{F}}$. So we have that $\mathcal{O}_{\mathcal{F}} = \text{Ker}(\mathcal{C}_M^\infty \xrightarrow{d_e} \underline{E}^{*1})$.

The sheaves \mathcal{C}_M^∞ , \underline{E}^{*s} being fine, our lemma will be proved once it is established that the complex (1.20) is exact.

The problem is local so we work on the open set $W = U \times V \subset \mathbb{R}^p \times \mathbb{C}^q$, where U is the unit ball and V is the unit polydisk. We assume that $\mathcal{F}|_W$ is given by the fibers of the projection $U \times V \rightarrow V$ and we let $(x, z) = (x^1, \dots, x^p, z^{p+1}, \dots, z^n)$ be the local coordinate functions. The complex $(\Gamma(W, \underline{E}^{*s}), d_e)$ is isomorphic to the double complex $(A^{s,t}, d_{\parallel} + \bar{\partial})$, $s, t \geq 0$, where $A^{s,t}$ is the space of $s + t$ -forms on W of the form $\varphi = \varphi_{I,J}(x, z)[dx^I] \wedge [d\bar{z}^J]$, $|I| = s$, $|J| = t$ and where

$$d_{\parallel} \varphi = \frac{\partial \varphi_{I,J}}{\partial x^i} [dx^i \wedge dx^I] \wedge [d\bar{z}^J],$$

$$\bar{\partial} \varphi = \frac{\partial}{\partial \bar{z}^\alpha} \varphi_{I,J} [d\bar{z}^\alpha] \wedge [dx^I] \wedge [d\bar{z}^J] = (-1)^s \frac{\partial \varphi_{I,J}}{\partial \bar{z}^\alpha} [dx^I] \wedge [d\bar{z}^\alpha] \wedge [d\bar{z}^J].$$

Consider the spectral sequence associated to the second filtration on A^{\bullet} . Then

$$(1.21) \quad {}''E_2^{s,t} \cong H^s(H^t(A^{\bullet}, d_{\parallel}), \bar{\partial}) \Rightarrow H^s(\Gamma(W, \underline{E}^*), d_e).$$

We must show that ${}''E_2^{s,t} = 0$ for $s + t > 0$. If we consider A^{\bullet} as the de Rham complex of U parametrized by V , the proof of the Poincaré lemma [10] goes through to show that this sequence collapses to ${}''E_2^{s,0} \cong H^s(V, \mathcal{O}_V)$,

where \mathcal{O}_V is the sheaf of germs of holomorphic functions on V . But by Dolbeaut's lemma [10], this cohomology group is trivial for $s > 0$, so we are done.

Lemma 1.22. *The complex (E^{**}, d_e) is elliptic.*

Proof. We show that if $x \in M$ and $\theta \in T_{M_x}^*$ is a co-vector at x , the symbol sequence

$$(1.23) \quad \mathbb{C} \xrightarrow{\sigma_\theta} E_x^* \xrightarrow{\sigma_\theta} \Lambda^2 E_x^* \rightarrow \dots$$

is exact. For $\beta \in \Lambda^s E_x^*$ let $\tilde{\beta} \in \Lambda^s(T_{M_x}^* \otimes \mathbb{C})$ be a form such that $i_E^*(\tilde{\beta}) = \beta$. It is easily seen that the symbol of d_e at $\theta \in T_{M_x}^*$ is given by

$$(1.24) \quad \sigma_\theta(\beta) = i_E^*(\theta \wedge \tilde{\beta}).$$

To prove exactness pick a basis for $T_{M_x}^* \otimes \mathbb{C}$ of the form $\theta, dz^1, \dots, dz^q, \xi_{q+2}, \dots, \xi_n$, and let $\beta \in E_x^*$ be such that $\sigma_\theta(\beta) = 0$. We will find $\tilde{\alpha} \in \Lambda^{s-1}(T_{M_x}^* \otimes \mathbb{C})$ for which $i_E^*(\theta \wedge \tilde{\alpha}) = \beta$. We proceed as follows: since $i_E^*(\theta \wedge \tilde{\beta}) = 0$ we can write $\theta \wedge \tilde{\beta}$ in the form $\theta \wedge \tilde{\beta} = dz^\alpha \wedge \gamma_\alpha$ where γ_α can be written in terms of $\theta, dz^{\alpha+1}, \dots, dz^q, \xi_{q+2}, \dots, \xi_n$ for $j = 1, \dots, q$. Since $\theta \wedge \theta \wedge \tilde{\beta} = 0$ we have $0 = \theta \wedge dz^\alpha \wedge \gamma_\alpha$, and since θ, dz^α, \dots is a basis it follows from the forms of the γ_α that $\theta \wedge \gamma_\alpha = 0$. Hence we can write $\gamma_\alpha = -\theta \wedge \delta_\alpha, \alpha = 1, \dots, q$. Let $\delta = dz^\alpha \wedge \delta_\alpha$, then $\theta \wedge \delta = \theta \wedge \tilde{\beta}$ so

$$(1.25) \quad \theta \wedge (\tilde{\beta} - \delta) = 0.$$

But $i_E^*(\delta) = 0$ hence

$$(1.26) \quad i_E^*(\tilde{\beta} - \delta) = \beta.$$

By (1.25) there is an element $\tilde{\alpha} \in \Lambda^{s-1}(T_{M_x}^* \otimes \mathbb{C})$ for which $\theta \wedge \tilde{\alpha} = (\tilde{\beta} - \delta)$. Hence by (1.26) we have $\sigma_\theta(\alpha) = i_E^*(\theta \wedge \tilde{\alpha}) = i_E^*(\tilde{\beta} - \delta) = \beta$, where $\alpha = i_E^*(\tilde{\alpha})$. Thus the sequence is exact and the complex is elliptic. *q.e.d.*

We now define the notion of holomorphic vector field on a holomorphic foliation. Locally a holomorphic vector field is a lift, via a submersion in $\mathcal{H}_{\mathcal{G}}$, of a holomorphic vector field on \mathbb{C}^q . More precisely let $U \subseteq M$ be an open set such that there is a submersion $f: U \rightarrow \mathbb{C}^q, f \in \mathcal{H}_{\mathcal{G}}$. Then we define $\theta_{\mathcal{G}|U}$ as the pull-back $f^*(\theta_{\mathbb{C}^q})$ of the sheaf of germs of holomorphic vector fields on \mathbb{C}^q .

Remark. By a holomorphic vector field on \mathbb{C}^q we mean a holomorphic section of the holomorphic tangent bundle.

Theorem 1.27. *The cohomology groups $H^i(M, \mathcal{O}_{\mathcal{G}})$ and $H^i(M, \theta_{\mathcal{G}})$ are finite dimensional.*

Proof. Since the resolution (1.20) is elliptic, it follows from the theory of elliptic complexes [14] that $H^i(M, \mathcal{O}_{\mathcal{G}})$ is finite dimensional. Similarly, to show that $H^i(M, \theta_{\mathcal{G}})$ is finite dimensional we will construct a resolution of $\theta_{\mathcal{G}}$ by an

elliptic complex. Let

$$(1.28) \quad E_Q^{*i} = E^{*i} \otimes_{\theta_{\mathcal{F}}} \theta_{\mathcal{F}} \cong E^{*i} \otimes_{\mathbb{C}} Q^{(1,0)},$$

and let $d_Q = d_E \otimes \text{id}$. Since d_E is elliptic, so is d_Q and the required resolution is

$$(1.29) \quad 0 \rightarrow \theta_{\mathcal{F}} \rightarrow E_Q^{*0} \xrightarrow{d_Q} E_Q^{*1} \rightarrow \dots$$

This concludes the proof. q.e.d.

In adapted coordinates the operator d_Q is given by the formula

$$(1.30) \quad d_Q \left(\varphi^\alpha \otimes \left[\frac{\partial}{\partial z^\alpha} \right] \right) = d_z \varphi^\alpha \otimes \left[\frac{\partial}{\partial z^\alpha} \right],$$

where φ^α is in E^{*s} .

Remarks. The above discussion is an adaptation to holomorphic foliations of cohomology theories for C^∞ -foliations as presented in [9]. See also [4], [5] and [6]. We summarize here results of theirs which we will need in §4, as they apply to a holomorphic foliation \mathcal{F} , considered as a C^∞ -foliation.

On the complex $\Lambda^* L^*$ is a differential d_{\parallel} , which in adapted coordinates takes the form

$$(1.31) \quad d_{\parallel} \varphi = \frac{\partial \varphi_I}{\partial x^j} [dx^j] \wedge dx^I,$$

where

$$(1.32) \quad \varphi = \varphi_I [dx^I].$$

Let $\underline{C}_{\mathcal{F}}^\infty$ be the sheaf of complex-valued C^∞ -functions, which are locally constant along the leaves of \mathcal{F} . $(\Lambda^* \underline{L}^*, d_{\parallel})$ is a resolution of this sheaf [9].

Let $\underline{Q}_{\mathcal{F}}, \underline{Q}_{\mathcal{F}}^{(1,0)}$, etc., denote the sheaves of sections of $Q, Q^{(1,0)}$, etc, which are locally constant along the leaves of \mathcal{F} . These are all modules over $\underline{C}_{\mathcal{F}}^\infty$, and tensoring over $\underline{C}_{\mathcal{F}}^\infty$ with $(\Lambda^* \underline{L}^*, d_{\parallel})$ gives resolutions of these sheaves. In particular [9]

$$(1.33) \quad H^*(M, Q^{(0,s)} \otimes Q_{\mathcal{F}}^{(1,0)}) \cong H^*(\Gamma(\underbrace{Q^{*(0,s)} \otimes Q_{\mathcal{F}}^{(1,0)}}_{\underline{C}_{\mathcal{F}}^\infty} \otimes \Lambda^* \underline{L}^*), d_{Q_{\parallel}}),$$

where $d_{Q_{\parallel}} = \text{id} \otimes d_{\parallel}$.

2. Infinitesimal deformations and the Spencer operator

As an application of Theorem (1.27) we will show that the space of infinitesimal deformations of a holomorphic foliation is finite dimensional. Let $\mathcal{P}_{\mathcal{F}}$ be the pseudogroup of local diffeomorphisms of M which preserve the

holomorphic foliation \mathcal{F} . Specifically, $g: U \rightarrow V$ is in $\mathcal{P}_{\mathcal{F}}$ if and only if for each submersion $f_{\alpha}: U_{\alpha} \rightarrow \mathbf{C}^q$ in $\mathcal{H}_{\mathcal{F}}$ with $V \cap U_{\alpha} \neq \emptyset$, the submersion $f_{\beta} = f_{\alpha} \circ g: U \rightarrow \mathbf{C}^q$ is in $\mathcal{H}_{\mathcal{F}}$, where $U_{\beta} = U_{\alpha} \cap g^{-1}(U_{\alpha})$. It follows from the definition of a Haefliger cocycle that near each point $x \in U_{\alpha} \cap U_{\beta}$ there is a unique local biholomorphism $\tilde{g}_{\beta\alpha}^s$ of \mathbf{C}^q with

$$(2.1) \quad f_{\alpha} \circ g = \tilde{g}_{\beta\alpha}^x \circ f_{\alpha}.$$

Now let $\eta_{\mathcal{F}}$ be the sheaf of local vector fields whose flows lie in $\mathcal{P}_{\mathcal{F}}$, and let \underline{L} be the sheaf of local vector fields in $L \subseteq T_M$. The following lemma follows easily from (2.1).

Lemma 2.2. *The exact sequence (1.4) induces an exact sequence*

$$(2.3) \quad 0 \rightarrow \underline{L} \rightarrow \eta_{\mathcal{F}} \xrightarrow{\tau} \theta_{\mathcal{F}} \rightarrow 0.$$

Theorem 2.4. *The space of infinitesimal deformations of the pseudogroup $\mathcal{P}_{\mathcal{F}}$ is finite dimensional.*

Proof. Since \underline{L} is a finite sheaf, it follows that $H^j(M, \underline{L}) = 0$ for $j \geq 1$. Using the long exact cohomology sequence associated to (2.3) we see that $H^1(M, \eta_{\mathcal{F}}) \simeq H^1(M, \theta_{\mathcal{F}})$. By Spencer [13] the space of infinitesimal deformations of $\mathcal{P}_{\mathcal{F}}$ is just $H^1(M, \eta_{\mathcal{F}})$, which is finite dimensional by Theorem (1.27). q.e.d.

We will now define a nonlinear first order partial differential operator

$$D: \text{Hom}(E, Q^{(1,0)}) \rightarrow \text{Hom}(\Lambda^2 E, Q^{(1,0)}),$$

whose linearization is d_Q . We will call this the Spencer operator associated to \mathcal{F} . This operator is defined in analogy with the operator $\bar{\partial} - [,]$ which is of fundamental importance in the study of deformations of complex structure on a complex manifold. For this see [8].

D will be of the form, $D = d_Q - [,]_Q$ where $[,]_Q$ is an operator to be defined below. We will show that

$$(2.5) \quad \text{Fol}(\mathcal{F}) = \{ \varphi \in \text{Hom}(E, Q^{(1,0)}): D\varphi = 0 \}.$$

In §3 we will show how to realize $\text{Fol}(\mathcal{F})$, via (2.5), as an analytic subspace of $H^1(M, \theta_{\mathcal{F}})$.

Remark 2.6. Unfortunately, our techniques work only if we assume that the splitting (1.9) is induced by a foliation \mathcal{F}^{\perp} transverse to \mathcal{F} . The foliation \mathcal{F}^{\perp} need not be holomorphic. We assume from this point on that \mathcal{F}^{\perp} is fixed and that $\rho: Q \rightarrow T_M$ is the tangent bundle to \mathcal{F}^{\perp} . We can therefore think of all bundles as sub-bundles of the tensor algebra bundle of T_M . An adapted coordinate system will now be a chart (x, z) in $\mathbf{R}^p \times \mathbf{C}^q$ such that the projections $\mathbf{R}^p \times \mathbf{C}^q \rightarrow \mathbf{C}^q$ are in $\mathcal{H}_{\mathcal{F}}$ and such that the leaves \mathcal{F}^{\perp} are locally given by the sets $\{x = \text{constant}\}$.

In these coordinates $d_Q: E_Q^{*s} \rightarrow E_Q^{*s+1}$ is given by the formula

$$(2.7) \quad d_Q \left(\varphi_{JB}^\alpha dx^J \wedge d\bar{z}^B \otimes \frac{\partial}{\partial z^\alpha} \right) = \left(\frac{\partial}{\partial x^i} \varphi_{JB}^\alpha dx^i + \frac{\partial}{\partial \bar{z}^\beta} \varphi_{JB}^\alpha d\bar{z}^\beta \right) \wedge dx^J \wedge d\bar{z}^B \otimes \frac{\partial}{\partial z^\alpha},$$

where, as usual, $dx^J = dx^{j_1} \wedge \cdots \wedge dx^{j_l}$, $d\bar{z}^B = d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_k}$ and let $l + k = s$.

We now define $[\cdot, \cdot]_Q$, as a map $[\cdot, \cdot]_Q: E_Q^{*r} \times E_Q^{*s} \rightarrow E_Q^{*r+s}$ as follows. Choose adapted coordinates, and let

$$\begin{aligned} \varphi &= \varphi_{JB}^\alpha dx^J \wedge d\bar{z}^B \otimes \frac{\partial}{\partial z^\alpha} \in E_Q^{*r}, \\ \psi &= \psi_{KG}^\alpha dx^K \wedge d\bar{z}^G \otimes \frac{\partial}{\partial z^\alpha} \in E_Q^{*s}. \end{aligned}$$

Then

$$(2.8) \quad [\varphi, \psi]_Q = \frac{1}{2r!s!} \left(\varphi_{JB}^\gamma \frac{\partial}{\partial z^\gamma} \psi_{KG}^\alpha + (-1)^{rs+1} \psi_{KG}^\gamma \frac{\partial \varphi_{JB}^\alpha}{\partial z^\gamma} \right) dx^J \wedge d\bar{z}^B \wedge dx^K \wedge d\bar{z}^G \otimes \frac{\partial}{\partial z^\alpha}.$$

We now give a precise statement and proof of (2.5).

Proposition 2.9. *Given $\varphi \in \text{Hom}(E, Q^{(1,0)})$, the distribution E_φ defines a holomorphic foliation if and only if $D\varphi = 0$.*

Proof. By the complex Frobenius theorem we must show that $[E_\varphi, E_\varphi] \subseteq E_\varphi$ if and only if $D\varphi = 0$.

Again we work in adapted coordinates. Suppose $D\varphi = 0$. Then we see by the definition of E_φ that E_φ is generated by the vector fields

$$X_i = \frac{\partial}{\partial x^i} + \varphi_i^\alpha \frac{\partial}{\partial z^\alpha}, \quad Y_{\bar{\beta}} = \frac{\partial}{\partial \bar{z}^\beta} + \varphi_{\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha},$$

where $\varphi = \varphi_i^\alpha dx^i \otimes \partial/\partial z^\alpha + \varphi_{\bar{\beta}}^\alpha d\bar{z}^\beta \otimes \partial/\partial z^\alpha$. We need only show that for all i, j, α, β the vector fields $[X_i, X_j]$, $[X_i, Y_{\bar{\alpha}}]$, and $[Y_{\bar{\alpha}}, Y_{\bar{\beta}}]$ lie in E_φ .

It follows from (2.7) and (2.8) that

$$(2.10) \quad \begin{aligned} [X_i, X_j] &= (D\varphi)_{ij}^\alpha \frac{\partial}{\partial z^\alpha}, \\ [X_i, Y_{\bar{\beta}}] &= (D\varphi)_{i\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha}, \\ [Y_{\bar{\beta}}, Y_{\bar{\gamma}}] &= (D\varphi)_{\bar{\beta}\bar{\gamma}}^\alpha \frac{\partial}{\partial z^\alpha}, \end{aligned}$$

where

$$D\varphi = (D\varphi)_{ij}^{\alpha} dx^i \wedge dx^j \otimes \frac{\partial}{\partial z^{\alpha}} + (D\varphi)_{i\bar{j}}^{\alpha} dx^i \wedge d\bar{z}^{\bar{j}} \otimes \frac{\partial}{\partial z^{\alpha}} + (D\varphi)_{\bar{i}\bar{j}}^{\alpha} d\bar{z}^{\bar{i}} \wedge d\bar{z}^{\bar{j}} \otimes \frac{\partial}{\partial z^{\alpha}}.$$

Hence all brackets are zero, and the distribution E_{φ} is involutive.

Conversely suppose E_{φ} is involutive. Then the brackets $[X_i, X_j]$, $[X_i, Y_{\bar{\alpha}}]$, $[Y_{\bar{\alpha}}, Y_{\bar{\beta}}]$ lie in E_{φ} . But by (2.10) this is impossible unless all brackets are zero. Again, by (2.10) this is impossible unless $D\varphi = 0$. q.e.d.

We conclude this section with a list of properties of $[,]_{\varrho}$ which will be needed in the construction of solutions of $D\varphi = 0$. They are easily verified.

(2.11) $[,]_{\varrho}$ is bilinear.

If $\varphi \in E_{\varrho}^{*r}$, $\psi \in E_{\varrho}^{*s}$ and $\tau \in E_{\varrho}^{*t}$, then

(2.12) $[\varphi, \psi]_{\varrho} = (-1)^{rs} [\psi, \varphi]_{\varrho}$,

(2.13) $d_{\varrho}[\varphi, \psi]_{\varrho} = [d_{\varrho}\varphi, \psi]_{\varrho} + (-1)^r [\varphi, d_{\varrho}\psi]_{\varrho}$,

(2.14) $(-1)^{rs} [\varphi, [\psi, \tau]_{\varrho}]_{\varrho} + (-1)^{st} [\psi, [\tau, \varphi]_{\varrho}]_{\varrho} + (-1)^{rt} [\tau, [\varphi, \psi]_{\varrho}]_{\varrho} = 0$.

3. The Kuranishi family of a holomorphic foliation

In this section we extend Kuranishi's theorem [8] on the existence of locally complete families of complex analytic structures to the case of a holomorphic foliation for which there is a transverse foliation. More specifically, we will prove the following theorem.

Theorem 3.1. *Let \mathcal{F}_0 be a holomorphic foliation on a compact C^{∞} -manifold M , and let \mathcal{F}^{\perp} be a C^{∞} -foliation transverse to \mathcal{F}_0 . Then there are a local analytic subset $B \subseteq H^1(M, \theta_{\mathcal{F}_0})$ and a holomorphic map*

(3.2) $B \rightarrow \mathcal{F}ol(\mathcal{F}_0) \subseteq \text{Hom}(E, Q^{(1,0)}): t \rightarrow \mathcal{F}_t,$

which defines a locally complete family of holomorphic foliations in the sense that if $\tilde{\mathcal{F}}$ is a holomorphic foliation sufficiently close to \mathcal{F}_0 , then $\tilde{\mathcal{F}}$ is conjugate to a foliation of the form \mathcal{F}_t via a diffeomorphism of M close to the identity. Furthermore, given a Riemannian metric respecting the local product structure on M induced by \mathcal{F}_0 and \mathcal{F}^{\perp} this diffeomorphism can be unambiguously defined.

Remarks 3.3. This theorem is a generalization of Kuranishi's theorem in the following sense. A complex manifold M can be thought of as the

holomorphic foliation on M given by points. The foliation \mathcal{F}^\perp is just the codimension -0 foliation of M whose single leaf is M itself.

The proof of Theorem 3.1 is an adaptation of Kuranishi's proof [8]. In fact, if the following substitutions are made, the proofs are almost identical: replace the Dolbault complex by (E_Q^*, d_Q) and replace the bracket operation of Kuranishi by $[\cdot, \cdot]_Q$. In place of the operator $\bar{\partial} - [\cdot, \cdot]$ substitute the operator $D = d_Q - [\cdot, \cdot]_Q$. The proof of Theorem 3.1 proceeds in two steps. We first construct the family \mathcal{F}_t as solutions of a certain system of equations. Then we show that any holomorphic foliation close to \mathcal{F}_0 is conjugate to \mathcal{F}_t for some t .

Step 1. The construction of \mathcal{F}_t . We will now construct a map from elements of a certain analytic subset B of $H^1(M, \theta_{\mathcal{F}})$ near zero to solutions of the system of equations

$$(3.4) \quad d_Q \varphi = [\varphi, \varphi]_Q, \quad \delta_Q \varphi = 0$$

with $\varphi \in \Gamma(E_Q^{*1}) \cong \text{Hom}(E, Q^{1,0})$ having small norm. Here δ_Q denotes the adjoint of the operator $d_Q: \Gamma(E_Q^{*s}) \rightarrow \Gamma(E_Q^{*s+1})$ with respect to the inner product induced by the Riemannian metric on M associated to an $SO(p) \times U(q)$ reduction of the tangent bundle of M which is compatible with the local product structure on M and the complex structure on Q .

Recall that, by the Hodge decomposition theorem for elliptic complexes [14], there is a Green's operator

$$(3.5) \quad G_Q: \Gamma(E_Q^{*r}) \rightarrow \Gamma(E_Q^{*r}), \quad r \geq 0$$

with the property that

$$(3.6) \quad I = H_Q + \Delta_Q \circ G_Q,$$

where $\Delta_Q = d_Q \delta_Q + \delta_Q d_Q$, and $H_Q: \Gamma(E_Q^{*r}) \rightarrow H^r(M, \theta_{\mathcal{F}})$ is projection onto $\text{Ker } \Delta_Q$, which by Lemma 1.19 we can identify with $H^r(M, \theta_{\mathcal{F}})$.

Let $\|\cdot\|_s$ denote the Sobolov norm on $H^*(M, \theta_{\mathcal{F}})$ induced by the metric on M . Pick a basis $\varphi_1, \varphi_2, \dots, \varphi_m$ for $H^1(M, \theta_{\mathcal{F}})$. Given $\varphi_0 = \sum_{i=1}^m t_i \varphi_i \in H^1(M, \theta_{\mathcal{F}})$ with $\|\varphi_0\|_s$ small, say $< \varepsilon$, we wish to solve the equation

$$(3.7) \quad \varphi = \varphi_0 + \varphi_Q G_Q [\varphi, \varphi]_Q,$$

and show that the solution $\varphi(t)$ depends holomorphically on $t = (t_1, \dots, t_m) \in \mathbb{C}^m$. To do this we need two estimates:

$$(3.8) \quad \|[\varphi_1, \varphi_2]_Q\|_s \leq C \|\varphi_1\|_{s+1} \cdot \|\varphi_2\|_{s+1},$$

and

$$(3.9) \quad \|\delta_Q G_Q \varphi\|_s \leq C \|\varphi\|_{s-1},$$

$$(3.10) \quad \|H_Q \varphi\|_s \leq C \|\varphi\|_s.$$

The first estimate follows trivially from the definition of $[\cdot, \cdot]_Q$, and the second

and third follow from the fact that the d_Q -complex is elliptic. The solution of (3.7) and its holomorphic dependence follow, verbatim as in [7] using the implicit function theorem or a power series expansion.

We can now solve the system (3.4) using the above result. Begin by assuming that φ is a solution of (3.7). We will soon see that for $\|\varphi\|_s$ sufficiently small this assumption is redundant. Note that, by the Hodge decomposition (3.6),

$$(3.11) \quad [\varphi, \varphi]_Q = H_Q[\varphi, \varphi]_Q + d_Q \delta_Q G_Q[\varphi, \varphi]_Q + \delta_Q d_Q G_Q[\varphi, \varphi]_Q,$$

and that, since $d_Q \varphi_0 = 0$,

$$(3.12) \quad d_Q \varphi = d_Q \delta_Q G_Q[\varphi, \varphi]_Q.$$

Combining (3.11) and (3.12) yields

$$-d_Q \varphi + [\varphi, \varphi]_Q = H_Q[\varphi, \varphi]_Q + \delta_Q d_Q G_Q[\varphi, \varphi]_Q,$$

and therefore

$$(3.13) \quad -d_Q \varphi + [\varphi, \varphi]_Q = H_Q[\varphi, \varphi]_Q + \delta_Q G_Q d_Q[\varphi, \varphi]_Q,$$

since $d_Q G_Q = G_Q d_Q$. Since the terms on the right are orthogonal, φ is a solution of (3.4) if and only if the equations

$$(3.14) \quad H_Q[\varphi, \varphi]_Q = 0,$$

$$(3.15) \quad \delta_Q G_Q d_Q[\varphi, \varphi]_Q = 0$$

are satisfied. However, (3.15) is a consequence of (3.14) by the following argument. First observe that

$$(3.16) \quad \delta_Q G_Q d_Q[\varphi, \varphi]_Q = 2\delta_Q G_Q[d_Q \varphi, \varphi]_Q$$

by (2.12) and (2.13). If $H_Q[\varphi, \varphi]_Q = 0$, then by (3.13) we can write (3.16) as

$$(3.17) \quad \begin{aligned} \delta_Q G_Q d_Q[\varphi, \varphi]_Q &= 2\delta_Q G_Q[[\varphi, \varphi]_Q, \varphi]_Q - 2\delta_Q G_Q[\delta_Q G_Q d_Q[\varphi, \varphi]_Q, \varphi]_Q \\ &= -2\delta_Q G_Q[\delta_Q G_Q d_Q[\varphi, \varphi]_Q, \varphi]_Q \end{aligned}$$

by the Jacobi identity (2.14). Hence by (3.8) and (3.9) we have the inequality

$$\|\delta_Q G_Q d_Q[\varphi, \varphi]_Q\|_s \leq C \|\delta_Q G_Q d_Q[\varphi, \varphi]_Q\|_s \|\varphi\|_s.$$

So, for $\|\varphi\|_s$ sufficiently small, (3.15) holds.

We can now construct the space B of the theorem. Let

$$(3.18) \quad B = \{ \varphi_0 \in H^1(M, \theta_{\overline{\varphi}}) \mid \|\varphi_0\| < \varepsilon, \quad H_Q[\varphi(t), \varphi(t)]_Q = 0 \},$$

where ε is to be chosen as in Lemma 3.23. This is an analytic subset of $H^1(M, \theta_{\overline{\varphi}})$. Furthermore, by the above argument, the elements $\varphi(t)$ for

$\sum t_i \varphi_i \in B$ are solutions of the equation $D\varphi \equiv d_Q \varphi - [\varphi, \varphi]_Q = 0$, and therefore define holomorphic foliations.

Note that if ψ is a solution of (3.4) of sufficiently small norm, then $\psi = \varphi(t)$ for a unique element $\varphi_0 = \sum t_i \varphi_i \in B$. To see this, notice that since $D\psi = 0$ and $\delta_Q \psi = 0$, we have

$$(3.19) \quad \Delta_Q \psi = \delta_Q[\psi, \psi]_Q.$$

Hence

$$(3.20) \quad \psi - H_Q \psi = G_Q \delta_Q[\psi, \psi]_Q.$$

Set $\varphi_0 = H_Q \psi$. Then from (3.20)

$$(3.21) \quad \psi = \varphi_0 + \delta_Q G_Q[\psi, \psi]_Q.$$

By (3.10)

$$(3.22) \quad \|\varphi_0\|_s = \|H_Q \psi\|_s \leq c \|\psi\|_s.$$

Therefore there is a number $\eta > 0$ with the property that if $\|\psi\|_s < \eta$, then $\|\varphi_0\|_s < \varepsilon$. Hence $\psi = \varphi(t)$ for $\varphi_0 = \sum t_i \varphi_i \in B$ by the following lemma.

Lemma 3.23. *The set $\{\varphi(t) | \sum t_i \varphi_i \in B\}$ comprises all solutions of (3.7) of small norm, and these solutions are unique.*

Proof. Fix φ_0 with $\|\varphi_0\|_s$ small, and let $\varphi(t)$ be the solution obtained by power series. Suppose ω is another solution. Let $\omega = \varphi - \varphi(t)$. Then

$$\begin{aligned} \omega &= \delta_Q G_Q([\varphi, \varphi]_Q - [\varphi(t), \varphi(t)]_Q) \\ &= \delta_Q G_Q([\omega, \varphi(t)]_Q + [\varphi(t), \omega]_Q + [\omega, \omega]_Q) \\ &= \delta_Q G_Q(2[\omega, \varphi(t)]_Q + [\omega, \omega]_Q). \end{aligned}$$

Hence by (3.8)

$$\|\omega\|_s \leq c \|\omega\|_s (\|\varphi(t)\|_s + \|\omega\|_s).$$

For $\|\varphi(t)\|_s$ sufficiently small say $< \varepsilon$, this can only happen if $\omega = 0$. *q.e.d.*

At this point we have shown that every solution of the equations $D\varphi = 0$ and $\delta_Q \varphi = 0$ is of the form $\varphi(t)$ for $\varphi_0 = \sum t_i \varphi_i \in B$.

Step 2. Suppose now that the norm of φ is small, and that $D\varphi = 0$, but that $\delta_Q \varphi \neq 0$. We wish to show that the corresponding foliation \mathcal{F}_φ is conjugate to one of the form $\mathcal{F}_{\varphi(t)}$ for $\sum t_i \varphi_i \in B$. As in Kuranishi [8], we do this using diffeomorphisms generated by geodesics.

We just examine the action of diffeomorphisms of M near the identity on holomorphic foliations, or more precisely their associated distributions. Let $\varphi \in \text{Hom}(E, Q^{(1,0)})$, and denote the distribution associated to φ by $E_\varphi \subseteq T_M^{\mathbb{C}}$. Let f be a diffeomorphism of M close to the identity in the C^∞ -topology. Then the Jacobian map f_* maps E_φ to a bundle $f_*(E_\varphi)$, and there is a unique

element $\psi \in \text{Hom}(E, Q^{(1,0)})$ with $E_\psi = f_*(E_\varphi)$. Denote this element by $f_*\varphi$.

We wish to find a formula for $f_*\varphi$ in terms of f and φ in adapted coordinates. Let $(x, z) = (x^1, \dots, x^p, z^{p+1}, \dots, z^n)$ be adapted coordinates. Then

$$(3.24) \quad \varphi = \varphi_i^\alpha dx^i \otimes \frac{\partial}{\partial z^\alpha} + \varphi_{\bar{\beta}}^\alpha d\bar{z}^\beta \otimes \frac{\partial}{\partial z^\alpha},$$

and E_φ is spanned locally by the vector fields

$$(3.25) \quad X_i^\varphi = \frac{\partial}{\partial x^i} + \varphi_i^\alpha \frac{\partial}{\partial z^\alpha}, \quad X_{\bar{\beta}}^\varphi = \frac{\partial}{\partial \bar{z}^\beta} + \varphi_{\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha}.$$

Hence locally $f_*(E_\varphi)$ is spanned by the entries of the $(p + q) \times 1$ matrix

$$(3.26) \quad \begin{bmatrix} f_*(X_i^\varphi) \\ f_*(X_{\bar{\beta}}^\varphi) \end{bmatrix} = \begin{pmatrix} M_{ij} & M_{i\alpha} \\ M_{\bar{\beta}j} & M_{\bar{\beta}\alpha} \end{pmatrix} \begin{bmatrix} \frac{\partial}{\partial x^j} \\ \frac{\partial}{\partial \bar{z}^\alpha} \end{bmatrix} + \begin{pmatrix} N_{i\alpha} \\ N_{\bar{\beta}\alpha} \end{pmatrix} \left(\frac{\partial}{\partial z^\alpha} \right),$$

where

$$(3.27) \quad \begin{aligned} M_{ij} &= \left(\frac{\partial f^j}{\partial x^i} + \varphi_i^\gamma \frac{\partial f^j}{\partial z^\gamma} \right), & M_{i\alpha} &= \left(\frac{\partial f^\alpha}{\partial x^i} + \varphi_i^\gamma \frac{\partial f^\alpha}{\partial z^\gamma} \right), \\ M_{\bar{\beta}j} &= \left(\frac{\partial f^j}{\partial \bar{z}^\beta} + \varphi_{\bar{\beta}}^\gamma \frac{\partial f^j}{\partial z^\gamma} \right), & M_{\bar{\beta}\alpha} &= \left(\frac{\partial f^\alpha}{\partial \bar{z}^\beta} + \varphi_{\bar{\beta}}^\gamma \frac{\partial f^\alpha}{\partial z^\gamma} \right), \\ N_{i\alpha} &= \left(\frac{\partial f^\alpha}{\partial x^i} + \varphi_i^\gamma \frac{\partial f^\alpha}{\partial z^\gamma} \right), & N_{\bar{\beta}\alpha} &= \left(\frac{\partial f^\alpha}{\partial \bar{z}^\beta} + \varphi_{\bar{\beta}}^\gamma \frac{\partial f^\alpha}{\partial z^\gamma} \right), \end{aligned}$$

and $f = (f^1, \dots, f^n)$. Setting

$$(3.28) \quad \psi = f_*(\varphi) = \psi_i^\alpha dx^i \otimes \frac{\partial}{\partial z^\alpha} + \psi_{\bar{\beta}}^\alpha d\bar{z}^\beta \otimes \frac{\partial}{\partial z^\alpha},$$

we see that $f_*(E_\varphi)$ is spanned locally by the vectors of the matrix

$$(3.29) \quad \begin{bmatrix} \frac{\partial}{\partial x^i} \\ \frac{\partial}{\partial \bar{z}^\beta} \end{bmatrix} + \begin{pmatrix} \psi_i^\gamma \\ \psi_{\bar{\beta}}^\gamma \end{pmatrix} \left(\frac{\partial}{\partial z^\gamma} \right).$$

Since f is near the identity, the matrix

$$(3.30) \quad M = \begin{pmatrix} M_{ij} & M_{i\alpha} \\ M_{\bar{\beta}j} & M_{\bar{\beta}\alpha} \end{pmatrix}$$

is invertible. Combining (3.26) and (3.29) we see that

$$(3.31) \quad \begin{pmatrix} \psi_i^\gamma \\ \psi_{\bar{\beta}}^\gamma \end{pmatrix} = M^{-1} \circ N,$$

where

$$(3.32) \quad N = \begin{pmatrix} N_{i\alpha} \\ N_{\bar{\beta}\alpha} \end{pmatrix}.$$

We summarize our results in the following lemma.

Lemma 3.33. *Let $\varphi \in \text{Hom}(E, Q^{(1,0)})$, and let f be a diffeomorphism of M near the identity in the Whitney C^∞ -topology. Then in adapted coordinates $\psi = f_*(\varphi)$ is given by (3.31).*

We will now apply Lemma 3.33 to diffeomorphisms associated to geodesics. Considering $Q^{(1,0)}$ as a real vector bundle, we see that the map τ of (1.4) induces an isomorphism $Q \xrightarrow{\tau} Q^{(1,0)}$. Use τ to identify $Q^{(1,0)}$ with $Q \subseteq T_M$. See [26]. Let $X \in \Gamma(Q^{(1,0)}) \subseteq \Gamma(T_M)$ be a vector field close to zero in the C^∞ -topology. Since M is compact, it is complete in our metric. Consider the map $f(X, \cdot): M \rightarrow M$ defined by

$$(3.34) \quad f(X, y) \equiv \gamma(X, y, 1),$$

where $t \rightarrow \gamma(X, y, t)$ is the geodesic with initial conditions

$$(3.35) \quad \gamma(X, y, 0) = y, \quad \gamma'(X, y, 0) = X(y).$$

For X small, $f(X, \cdot)$ is a diffeomorphism of M . We wish to express $f(X, \cdot)$ locally as a Taylor series in the components of X , and use this expansion to represent (3.31) in terms of the components of X . In adapted coordinates $f(X, x, z) = (f^j(X, x, z), f^\alpha(X, x, z))$ and since $f(tX, x, z) = \gamma(X, (x, z), t)$ the equations

$$(3.36) \quad \begin{aligned} X^\alpha \frac{\partial f^j}{\partial X^\alpha}(0, x, z) + \bar{X}^\alpha \frac{\partial f^j}{\partial \bar{X}^\alpha}(0, x, z) &= \frac{d}{dt} \gamma^j(X, (x, z), 0) = 0, \\ X^\alpha \frac{\partial f^\beta}{\partial X^\alpha}(0, x, z) + \bar{X}^\alpha \frac{\partial f^\beta}{\partial \bar{X}^\alpha}(0, x, z) &= \frac{d}{dt} \gamma^\beta(X, (x, z), 0) = X^\beta \end{aligned}$$

are satisfied, where $X = X^\alpha \partial / \partial z^\alpha$. Therefore

$$(3.37) \quad \frac{\partial f^j}{\partial X^\alpha} = \frac{\partial f^j}{\partial \bar{X}^\alpha} = \frac{\partial f^\beta}{\partial \bar{X}^\alpha} = 0,$$

$$(3.38) \quad \frac{\partial f^\beta}{\partial X^\alpha} = \delta_\alpha^\beta.$$

Hence f is of the form

$$(3.39) \quad \begin{aligned} f^j(X, x, z) &= x^j + X^\alpha X^\beta r_{\alpha\beta}^j(X, x, z), \\ f^\alpha(X, x, z) &= z^\alpha + X^\alpha + X^\beta X^\gamma r_{\beta\gamma}^\alpha(X, x, z). \end{aligned}$$

Now for X close to zero, the matrix M can be written in the form $I + A_{tX}$, where $A_{tX} = t\tilde{A}_{(X,t)}$ and $\tilde{A}_{(X,t)}$ is a matrix-valued C^∞ -function in X^α ,

$\partial X^\alpha / \partial x^i, \partial X^\alpha / \partial z^\beta, \partial X^\alpha / \partial \bar{z}^\beta, \varphi_i^\alpha, \varphi_\beta^\alpha$ and t . Hence

$$(3.40) \quad M_{tX, \varphi}^{-1} = \sum_{l=0}^{\infty} (-1)^l A_{tX, \varphi}^l = I + H_{tX, \varphi},$$

where H is C^∞ in the variables $X^\alpha, \partial X^\alpha / \partial x^i$, etc. Also N can be expressed in the form

$$(3.41) \quad N_{tX} = \begin{pmatrix} \varphi_i^\alpha \\ \varphi_\beta^\alpha \end{pmatrix} + t \begin{pmatrix} \frac{\partial X^\alpha}{\partial x^i} \\ \frac{\partial X^\alpha}{\partial \bar{z}^\beta} \end{pmatrix} + tK_{tX, \varphi},$$

where $K_{tX, \varphi}$ is C^∞ in the variables $X^\alpha, \partial X^\alpha / \partial x_i$, etc. (3.40) and (3.41) allow us to write (3.31) in the form:

$$(3.42) \quad \begin{pmatrix} \psi_i^\alpha \\ \psi_\beta^\alpha \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x^i}(X^\alpha) \\ \frac{\partial}{\partial \bar{z}^\beta}(X^\alpha) \end{pmatrix} + \begin{pmatrix} \varphi_i^\alpha \\ \varphi_\beta^\alpha \end{pmatrix} + R(X, \varphi),$$

where $R(tX, t\varphi) = t^2 R_1(X, \varphi, t)$, and R_1 is C^∞ in t, φ, X and their derivatives. In invariant form, (3.42) reads

$$(3.43) \quad f_* \varphi = d_Q X + \varphi + R(\psi, X),$$

where $R(t\psi, tX) = t^2 R_1(\psi, X, t)$, and R_1 is C^∞ in t, X, φ and their derivatives.

We will now use (3.43) to show that if $\varphi \in \text{Hom}(E, Q^{(1,0)})$ is a solution of the equation $D\varphi = 0$ with $\|\varphi\|_s$ sufficiently small, then there is a unique element $\varphi(t)$ with $\sum t_i \varphi_i \in B$ and a unique vector field $X \in \Gamma(Q^{(1,0)})$ with $f_*(X, \cdot)(\varphi(t)) = \varphi$. This will complete Step 2 and the proof of Theorem 3.1.

Proposition 3.44. *Let H^\perp be the orthogonal complement of the space $\Gamma(\theta_\varphi)$ of \mathbb{F} -invariant holomorphic vector fields in $\Gamma(Q^{1,0})$. Then there is a neighborhood U of the origin of H^\perp and a neighborhood V_1 of the origin of $\Gamma(E_Q^{*1}) = \text{Hom}(E, Q^{(1,0)})$ such that for any element $\varphi \in V$ satisfying the equation $D\varphi = 0$, there is a unique element $X \in U$ with $f_*(X, \cdot)\varphi = \varphi(t)$ for $\sum t_i \varphi_i \in B$.*

Proof. Set $f = f(X, \cdot)$. Then $f_* \varphi$ is of the required form, provided only that $\delta_Q(f_* \varphi) = 0$. This follows from Step 1. But $\delta_Q(f_* \varphi) = 0$ if and only if

$$(3.45) \quad \delta_Q d_Q X + \delta_Q \varphi + \delta_Q R(\varphi, X) = 0$$

by (3.43). Since $X \in H^\perp$ it satisfies the equation

$$(3.46) \quad X = G_Q \Delta_Q X \equiv G_Q \delta_Q d_Q X.$$

Hence $\delta_Q(f_*\varphi) = 0$ if and only if

$$(3.47) \quad G_Q(\delta_Q d_Q X + \delta_Q \varphi + \delta_Q R(\varphi, X)) = 0,$$

or

$$(3.48) \quad X + G_Q \delta_Q \varphi + G_Q \delta_Q R(\varphi, X) = 0.$$

We will use the implicit function theorem to find such an X . Define a map

$$(3.49) \quad h: U_1 \times V_1 \subseteq H^\perp \times \Gamma(E_Q^{*1}) \rightarrow H^\perp$$

by

$$h(X, \varphi) = X + G_Q \delta_Q \varphi + G_Q \delta_Q R(\varphi, X),$$

where U_1 and V_1 have been chosen so that R is defined. If U_1, V_1 and H^\perp are given the topology induced by the Sobolev norm, then h is continuous and the Frechet derivative $\partial h / \partial X|_{(0,0)}$ is the identity map. Hence, by the implicit function theorem, there is a C^∞ -function $g: V \rightarrow U$ such that (3.48) holds if and only if $X = g(\varphi)$ for $\varphi \in V$. To see that X is smooth, note that it satisfies the second order elliptic equation with C^∞ coefficients

$$\Delta_Q X + \delta_Q R(\varphi, X) + \delta_Q \varphi = 0.$$

Hence X is smooth by the regularity theorem.

4. Computation of $H^*(M, \theta_{\mathfrak{F}})$

We now investigate the cohomology groups $H^*(M, \theta_{\mathfrak{F}})$. We begin by defining a filtration on the complex (1.28).

Let $Q^{(p,q)*}$ denote the sheaf of germs of sections of the bundle $\Lambda^p Q^{(1,0)*} \otimes \Lambda^q Q^{(1,0)*}$. Then the differential complex (1.28) is filtered as follows. For $s \geq 0$ let

$$(4.1) \quad F^s E_Q^{**} = \underline{Q}^{(0,s)*} \wedge \underline{E}_Q^{**s}.$$

Observe that $d_Q(F^s E_Q^{**}) \subset F^s E_Q^{**}$, as can easily be seen from the formulas (1.16) and (1.30) for d_e and d_Q . Associated to this filtration is a spectral sequence converging to $H^*(M, \theta_{\mathfrak{F}})$. The edge terms of this spectral sequence are of particular interest to us. Let $E_{Q,\mathfrak{F}}^{**s}$ be the subsheaf of \underline{E}_Q^{**s} consisting of sections which in adapted coordinates are of the form

$$(4.2) \quad \varphi = \varphi_{\beta,\alpha}(z) d\bar{z}^\beta \otimes \left[\frac{\partial}{\partial z^\alpha} \right].$$

Such sections are invariant under Lie differentiation with respect to vector fields tangent to \mathfrak{F} and are therefore called \mathfrak{F} -invariant sections. The restriction of d_Q to $E_{Q,\mathfrak{F}}^{**s}$ is denoted by \bar{d} and applied to a section φ as in (4.2) is of

the form

$$(4.3) \quad \bar{\partial}\varphi = \frac{\partial\varphi_{\beta,\alpha}}{\partial\bar{z}^\gamma} d\bar{z}^\gamma \wedge dz^\beta \otimes \left[\frac{\partial}{\partial z^\alpha} \right].$$

Clearly $\bar{\partial}(E_{Q,\mathfrak{F}}^{*\bullet}) \subseteq E_{Q,\mathfrak{F}}^{*\bullet}$ and there is a complex

$$(4.4) \quad 0 \rightarrow \underline{Q}_0^{(1,0)} \rightarrow E_{Q,\mathfrak{F}}^{*0} \xrightarrow{\bar{\partial}} E_{Q,\mathfrak{F}}^{*1} \xrightarrow{\bar{\partial}} \dots$$

Since $E = L^C \oplus Q^{(1,0)}$, there is an exact sequence

$$0 \rightarrow Q^{(0,1)*} \rightarrow E^* \rightarrow L^{C*} \rightarrow 0,$$

which induces exact sequences

$$(4.5) \quad 0 \rightarrow F^{p+1}E_{Q,\mathfrak{F}}^{*\bullet} \rightarrow F^pE_{Q,\mathfrak{F}}^{*\bullet} \xrightarrow{\tau} \Lambda^{*+p}L^{C*} \otimes_{C^{\infty}} Q^{(0,p)*} \otimes_{\mathfrak{F}} Q^{(1,0)} \rightarrow 0.$$

Now $\tau^*d_Q = d_{Q||} \tau$, hence by (1.33) we get the following result.

Lemma 4.6. $H^*(gr^p(E_{Q,\mathfrak{F}}^{*\bullet}), gr(d_Q)) \simeq H^*(M, Q^{(0,p)*} \otimes Q_{\mathfrak{F}}^{(1,0)})$.

The next proposition follows from (4.4) and (4.6).

Proposition 4.7. *The spectral sequence induced by the filtration F^{\bullet} of $E_{Q,\mathfrak{F}}^{*\bullet}$ converges to $H^*(M, \theta_{\mathfrak{F}})$. More specifically, $E_1^{s,t} = (H^t(M, Q^{(0,s)*} \otimes Q_{\mathfrak{F}}^{(1,0)})) \Rightarrow H^{s+t}(M, Q_{\mathfrak{F}}^{(1,0)})$ and $E_2^{s,0} = H^s(\Gamma(E_{Q,\mathfrak{F}}^{*\bullet}, \bar{\partial}))$.*

Recall that a V -manifold is an analytic space which locally has the structure of the orbit space defined by a finite group action on an open disc in \mathbb{C}^q where the group acts by biholomorphisms. By [3], if \mathfrak{F} is a Hausdorff foliation, the leaf space M/\mathfrak{F} has the structure of a V -manifold of the complex dimension q of the normal bundle to \mathfrak{F} . For details concerning V -manifolds, see Satake [12]. In case \mathfrak{F} has no holonomy, then M/\mathfrak{F} is non-singular and $M \rightarrow M/\mathfrak{F}$ is a fibration. A V -manifold N has a Dolbeaut complex defined on it and a holomorphic tangent bundle θ_N . Bailey [1] has shown that the cohomology groups $H^*(N, \theta_N)$ are finite dimensional. From the definition of the holomorphic tangent bundle of a V -manifold we have the following proposition.

Proposition 4.8. *If \mathfrak{F} is Hausdorff, then $E_2^{s,0} \cong H^s(M/\mathfrak{F}, \theta_{M/\mathfrak{F}})$ and this space is finite dimensional. Furthermore, if S denotes the generic leaf of \mathfrak{F} and $H^1(S, \mathbb{R}) = 0$, then $H^1(M, \theta_{\mathfrak{F}}) \cong H^1(M/\mathfrak{F}, \theta_{M/\mathfrak{F}})$.*

Proof. The first part of the proposition is immediate from the definitions.

To prove the second part of the proposition observe that

$$E_1^{0,1} = H^1(M, Q_{\mathfrak{F}}^{(1,0)}) = H^1(M, Q_{\mathfrak{F}}),$$

since $Q \cong Q^{(1,0)}$ by (1.2) and (1.4). Since Hamilton [4] has shown that $H^1(S, \mathbb{R}) = 0$ implies that $H^1(M, Q_{\mathfrak{F}}) = 0$, we have $E_2^{0,1} = 0$ and $E_2^{1,0} \cong H^1(M/\mathfrak{F}, \theta_{M/\mathfrak{F}})$ from which the result follows.

At this point we wish to present some cases where the groups $H^*(M, \theta_{\mathcal{F}})$ can be computed explicitly. The computations use standard techniques in sheaf theory and are quite similar to those of Mostow [9]. Therefore we will be brief.

We begin by considering the trivial example of a product foliation. Suppose that N is a complex manifold and that K is a compact C^∞ -manifold with $\dim_{\mathbb{C}} N = q$, $\dim_{\mathbb{R}} K = p$. Now let $M = N \times K$ and define \mathcal{F} to be the foliation on M given by the fibers of the projection $M \xrightarrow{\pi} N$. Then $\theta_{\mathcal{F}} = \pi^*(\theta_N)$. By Bredon [2] we obtain the next lemma.

Lemma 4.9. $H^*(M, \theta_{\mathcal{F}}) \cong H^*(N, \theta_N) \otimes_{\mathbb{C}} H_{DR}^*(K; \mathbb{C})$. In particular, if N is Stein $H^*(M, \theta_{\mathcal{F}}) \cong \Gamma(N, \theta_N) \otimes_{\mathbb{C}} H_{DR}(K, \mathbb{C})$.

If N is compact this implies the following corollary.

Corollary 4.10. *The set of holomorphic foliations near the holomorphic foliation \mathcal{F} , given as above, is a local analytic subset of the complex vector space $H^1(N, \theta_N) \oplus H_{DR}^1(K, \mathbb{C}) \otimes \Gamma(N, \theta_N)$.*

Assume that M is compact and that \mathcal{F} is a Hausdorff holomorphic foliation transverse to the fibers of a fibration $N \rightarrow M \rightarrow X$. Then N is a compact complex manifold and $M \simeq \tilde{X} \times N/G$, where \tilde{X} is a finite cover of X and $\tilde{M} = \tilde{X} \times N$ is a G manifold for G a finite group of deck transformations of \tilde{M} which acts biholomorphically on N . Further, \mathcal{F} is the foliation $\tilde{\mathcal{F}}/G$ for $\tilde{\mathcal{F}}$ the product foliation $\tilde{X} \times N \rightarrow N$. In this case G acts on $H^*(N, \theta_N)$ and on $H_{DR}^*(\tilde{X}, \mathbb{C})$ and we have the following proposition.

Proposition 4.11. $H^*(M, \theta_{\mathcal{F}}) \cong H_{DR}^*(\tilde{X}, \mathbb{C})^G \otimes_{\mathbb{C}} H^*(N, \theta_N)^G$ where $()^G$ denotes the space of G -invariant elements.

Proof. Consider the resolution (1.29) applied to $\tilde{\mathcal{F}}$ on \tilde{X} , i.e.,

$$0 \rightarrow \tilde{Q}^{(1,0)} \rightarrow \tilde{E}_Q^0 \xrightarrow{d_{\tilde{Q}}} \tilde{E}_Q^1 \rightarrow \dots$$

Then since G is finite $H^*(\Gamma(\tilde{E}_Q)^G, d_{\tilde{Q}}) = H^*(\Gamma(\tilde{E}_Q), d_{\tilde{Q}})^G$, and $\Gamma(\tilde{E}_Q)^G$ is isomorphic to the complex

$$0 \rightarrow \Gamma(\theta_{\mathcal{F}}) \rightarrow \Gamma(E_Q^0) \rightarrow \dots$$

associated to the resolution of $\theta_{\mathcal{F}}$. Therefore

$$H^*(M, \theta_{\mathcal{F}}) \cong H^*(\Gamma(\tilde{E}_Q)^G, d_{\tilde{Q}}) \simeq H^*(M, \theta_{\mathcal{F}})^G.$$

Note the above computation applies to the case where \mathcal{F} is given by the suspension via a biholomorphism $\varphi: N \rightarrow N$, where N is a compact complex manifold and φ has finite period.

References

- [1] W. Bailey, *The decomposition theorem for V manifolds*, Amer. J. Math. **78** (1956) 862–888.
- [2] G. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967.
- [3] D. B. A. Epstein, *Foliations with all leaves compact*, Dynamical Systems-Warwick, Lecture Notes in Math. Vol. 468, Springer, Berlin, 1974.
- [4] R. S. Hamilton, *Deformation theory for foliations*, preprint.
- [5] J. L. Heitsch, *A cohomology for foliated manifolds*, Comment. Math. Helv. **50** (1975) 197–218.
- [6] F. Kamber & Ph. Tondeur, *Invariant differential operators and the cohomology of Lie algebra sheaves*, Mem. Amer. Math. Soc. No. 113, 1971.
- [7] K. Kodaira, L. Nirenberg & D. Spencer, *On the existence of deformations of complex analytic structures*, Ann. of Math. **68** (1958) 450–459.
- [8] M. Kuranishi, *New proof for the existence of locally complete families of complex analytic structures*, Proc. Conf. Complex Analysis (Minneapolis), Springer, Berlin, 1965, 142–154.
- [9] M. Mostow, *Continuous cohomology of spaces with two topologies*, Mem. Amer. Math. Soc. **7** (1976) 1975.
- [10] R. Narasimhan, *Analysis on real and complex manifolds*, North Holland, Amsterdam, 1968.
- [11] L. Nirenberg, *A complex Frobenius theorem*, Seminar on Analytic Functions, Institute for Advanced Study, Princeton, 1957, 172–189.
- [12] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956) 359–363.
- [13] D. C. Spencer, *Deformations of structures on manifolds defined by transitive continuous pseudogroups*. I, II, Ann. of Math. **76** (1962) 306–445.
- [14] R. O. Wells, Jr., *Differential analysis on complex manifolds*, Prentice-Hall, Englewood Cliffs, N. J. 1973.
- [15] I. Vaisman, *Variétés Riemannienne feuilletées*, Czechoslovak Math. J. **21** (1971) 46–75.

UNIVERSITY OF UTAH